

# Semiclassical expansion of the ground state for a model of interacting spins in QED.

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## Abstract

In this article, we consider fixed spin-1/2 particles interacting through the quantized electromagnetic field in a constant magnetic field. We give some asymptotic expansions for the ground state and the ground state energy of the Hamiltonian operator  $H(h)$  describing this system. The first terms of these expansions enable to recover elementary formulas for the energy and the magnetic field of the spins when considered as magnets. A first order radiative correction is computed for the energy.

*Keywords:* Semiclassical analysis, spins interaction, quantum electrodynamics, quasimodes, ground state.

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## 1 The model.

The aim of this work is to give semiclassical expansions for the ground state and ground state energy for a Hamiltonian operator  $H(h)$  modelling the interaction between quantized electromagnetic field and  $N$  fixed spin-1/2 particles in a constant magnetic field.

We shall use a Hamiltonian operator  $H(h)$  recalled below in (10) and (11) (see Reuse[13], Hübner-Spohn [10], Dereziński-Gérard [7]).

The Hilbert space associated with this Hamiltonian is the completed tensor product  $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$ . The Hilbert space  $\mathcal{H}_{ph}$  for photons may be viewed as the symmetrized Fock space  $\mathcal{F}_s(H_{\mathbb{C}})$  associated with the complexified of some real Hilbert space  $H$  inspired by Lieb-Loss [11]. This space  $H$  is the space of mappings  $f = (f_1, f_2, f_3)$  from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  with  $f_j$  belonging in  $L^2(\mathbb{R}^3)$ , taking real values and satisfying,

$$k_1 f_1(k) + k_2 f_2(k) + k_3 f_3(k) = 0 \quad \text{a.e.} \quad (1)$$

This space is equipped with the norm,

$$|f|^2 = \sum_{j=1}^3 \int_{\mathbb{R}^3} |f_j(k)|^2 dk. \quad (2)$$

The Fock space  $\mathcal{F}_s(H_{\mathbb{C}})$  definition is reminded in Section 2. The space  $\mathcal{H}_{sp}$  for the spin particles is denoted by  $(\mathbb{C}^2)^{\otimes N}$ .

In the space  $\mathcal{H}_{ph}$ , the definition of the model and of the observables involves three kinds of operators: the number operator  $N$ , the free photons energy operator  $H_{ph}$  and operators at each point  $x \in \mathbb{R}^3$  associated with the three components of the magnetic field. These operators are denoted by  $B_m(x)$ ,  $1 \leq m \leq 3$  and for the electric field, it is denoted by  $E_m(x)$ ,  $1 \leq m \leq 3$ . Each of these operators is depending on the semiclassical parameter  $h > 0$  which is sometimes not explicitly written.

Within the Fock space formalism, the number operator  $N$  and the free photons Hamiltonian  $H_{ph}$  are defined by,

$$N = d\Gamma(I), \quad H_{ph} = h d\Gamma(M), \quad (3)$$

$M$  being the multiplication operator by  $\omega(k) = |k|$  with domain  $D(M) \subset H$ ,  $d\Gamma$  is the standard operator (see [12]) and  $h > 0$  is the semiclassical parameter. These equalities classically define selfadjoint operators (see [12]).

In the Fock space formalism, the operators  $B_m(x)$  (depending on  $h > 0$ ), is defined by,

$$B_m(x) = \sqrt{h} \Phi_S(a_m(x) + ib_m(x)) \quad (4)$$

where, for each  $a + ib \in H_{\mathbb{C}}$ ,  $\Phi_S(a + ib)$  is the Segal field, defined in [12], and  $a_m(x)$  and  $b_m(x)$  are elements of  $H$ , therefore mappings from  $\mathbb{R}^3$  into itself, defined by,

$$a_m(x)(k) = \frac{\chi(|k|)|k|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} \sin(k \cdot x) \frac{k \wedge e_m}{|k|} \quad (5)$$

$$b_m(x)(k) = \frac{\chi(|k|)|k|^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} \cos(k \cdot x) \frac{k \wedge e_m}{|k|}, \quad (6)$$

where  $\chi$  is a function belonging to  $\mathcal{S}(\mathbb{R})$  and  $(e_1, e_2, e_3)$  is the canonical basis of  $\mathbb{R}^3$ .

The following estimate will be used:

$$\|\Phi_S(a + ib)f\|^2 \leq 2(|a|^2 + |b|^2) \left[ \|f\|^2 + \langle Nf, f \rangle \right]. \quad (7)$$

Operators in  $\mathcal{H}_{sp}$  use in particular Pauli matrices  $\sigma_j$  ( $1 \leq j \leq 3$ ),

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (8)$$

For all  $\lambda \leq N$  and for any  $j \leq 3$ ,  $\sigma_j^{[\lambda]}$  denotes the following operator in  $\mathcal{H}_{sp}$ ,

$$\sigma_j^{[\lambda]} = I \otimes \cdots \otimes \sigma_j \otimes \cdots \otimes I, \quad (9)$$

where  $\sigma_j$  is located at the  $\lambda^{th}$  position.

We assume that there are  $N$  fixed spin-1/2 particles at points  $x_\lambda$  in  $\mathbb{R}^3$  ( $1 \leq \lambda \leq N$ ). Denoting the constant magnetic field by  $\beta = (\beta_1, \beta_2, \beta_3)$ , the system constituted with these particles and the quantized magnetic field is governed by the operator in  $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$  defined by,

$$H(h) = H_0 + hH_{int}, \quad H_0 = H_{ph} \otimes I, \quad (10)$$

where

$$H_{int} = \sum_{\lambda=1}^N \sum_{j=1}^3 (\beta_j + B_j(x_\lambda)) \otimes \sigma_j^{[\lambda]}. \quad (11)$$

It is recalled in [3] (Section 4) that it defines a selfadjoint operator with domain  $D(H_{ph}) \otimes \mathcal{H}_{sp}$ . In [3] some results of evolution are given, using the pseudodifferential calculus introduced in [1] and [2].

It is proved in [5], see also [4][8][9][10], that  $E_h$ , the infimum of the spectrum of the operator  $H(h)$ , is an eigenvalue and the associated eigenspace is of multiplicity 1.

**Theorem 1.1.** (i) If  $\chi$  is vanishing in a neighborhood of the origin, and if  $\beta \neq 0$ , one can find a sequence of real numbers  $\lambda_j$  such that, for all  $p$ ,

$$\left| E_h - \sum_{j=1}^p \lambda_j h^j \right| \leq C_p h^{p+1}. \quad (12)$$

(ii) If  $\beta \neq 0$ , but without the hypothesis on  $\chi$ , one can find  $\lambda_1$  and  $\lambda_2$  such that,

$$|E_h - \lambda_1 h - \lambda_2 h^2| \leq Ch^{5/2}. \quad (13)$$

The expansion is formally obtained in Proposition 2.3 and the control of the remainder term is derived in Theorem 2.7. Point (ii) is proved at the end of Section 2. For the first two terms, one finds,

$$\lambda_1 h + \lambda_2 h^2 = -N h |\beta| - \frac{h^2}{2} \sum_{\lambda, \mu \leq N} F(x_\lambda - x_\mu) - N C h^2,$$

where  $F$  is the semiclassical interaction function between parallel spins, given by, when the spins are aligned along the direction  $\mathbf{n}_\beta = \beta/|\beta|$ ,

$$F(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(|k|)|^2 \cos(k \cdot x) \left(1 - \frac{|k \cdot \mathbf{n}_\beta|^2}{|k|^2}\right) dk \quad (14)$$

and  $C$  is defined in (51). The first term amounts to the sum of the energies of each spin aligned along the direction of the constant field  $\beta$ . The second term amounts to the sum of the classical interaction energies between two spins, all being parallel (including for the auto-interaction). See the comments after Proposition 2.5. Only the third term  $N C h^2$  is genuinely a quantum term.

Now assuming that  $\chi$  is vanishing in a neighborhood of the origin, it is proved in Theorem 2.2 that a unitary eigenvector  $\varphi_h$  has, up to a normalization factor, an asymptotic expansion in powers of  $h^{1/2}$ , to any order. Without this hypothesis, we can give only an expansion with only three terms.

For all  $x \in \mathbb{R}^3$ , we can compare the average magnetic field  $\langle \mathbf{B}(x) \varphi_h, \varphi_h \rangle$  taken on the ground state  $\varphi_h$  with the magnetic field  $\mathbf{B}^{class}(x)$  associated by elementary physics with the spins systems regarded as magnets all being aligned along the direction of the (non zero) constant magnetic field. Setting  $\mathbf{n}_\beta = \beta/|\beta|$ , we have to consider the current density  $\mathbf{j}(x)$  corresponding to this spins system,

$$\mathbf{j}(x) = h \mathbf{n}_\beta \wedge \text{grad} \Phi(x), \quad (15)$$

$$\Phi(x) = \sum_{\lambda=1}^N \rho(x - x_\lambda), \quad \rho(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(k)|^2 \cos(k \cdot x) dk. \quad (16)$$

The potential vector  $\mathbf{A}^{class}(x)$  satisfies  $\Delta \mathbf{A}^{class} = \mathbf{j}$  and also,

$$\mathbf{A}^{class}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{j}(y)}{|x - y|} dy.$$

The classical magnetic field is,

$$\mathbf{B}^{class} = \text{rot} \mathbf{A}^{class}$$

and the electric field  $\mathbf{E}^{class}$  is zero. One notes the role of the function  $\chi$ : the spin particle is not exactly a point-like particle.

We shall prove the following theorem.

**Theorem 1.2.** *Suppose that  $\chi(0) = 0$  and  $\beta \neq 0$ . For all  $x$  in  $\mathbb{R}^3$ , we have*

$$\begin{aligned} \langle \mathbf{B}(x)\varphi_h, \varphi_h \rangle &= -\mathbf{B}^{class}(x) = \mathcal{O}(h^{3/2}), \\ \langle \mathbf{E}(x)\varphi_h, \varphi_h \rangle &= -\mathbf{E}^{class}(x) = \mathcal{O}(h^{3/2}). \end{aligned}$$

The first step of the proof is Proposition 2.6. The second step appears at the end of Section 2. The proof only used the fact that  $\chi(0) = 0$  and not that  $\chi$  is vanishing in a neighborhood of the origin. Nevertheless, the method used for these estimates, which relies on a conjugate operator, does not allow to avoid the hypothesis  $\chi(0) = 0$ .

## 2 Asymptotic expansions for the ground state.

The following theorem is proved in [5]. Let  $b_0$  and  $b_1$  be unitary elements of  $\mathbf{C}^2$  such that:

$$\sum_{m=1}^3 \beta_m \sigma_m b_0 = -|\beta| b_0 \quad \sum_{m=1}^3 \beta_m \sigma_m b_1 = |\beta| b_1$$

For all  $E \subset \{1, \dots, N\}$ ,  $a_E$  denotes the following element,

$$a_E = a_1 \otimes \dots \otimes a_N, \quad a_j = \begin{cases} b_1 & \text{if } j \in E \\ b_0 & \text{if } j \notin E \end{cases}. \quad (17)$$

**Theorem 2.1.** (*[4][5][8][9][10]*) *The infimum  $E_h$  of the spectrum of  $H(h)$  satisfies*

$$E_h \leq -N|\beta|h. \quad (18)$$

*The eigenspace associated with  $E_h$  has dimension 1. Moreover, there exists an unitary eigenvector  $\varphi_h$  corresponding to the eigenvalue  $E_h$ , the infimum of the spectrum of  $H(h)$ , such that, for any small enough  $h$ ,*

$$\|\varphi_h - (\Psi_0 \otimes a_\emptyset)\| \leq Ch^{1/2}. \quad (19)$$

The estimate (18) follows from  $\langle H(h)(\Psi_0 \otimes a_\emptyset), (\Psi_0 \otimes a_\emptyset) \rangle = -N|\beta|h$ . The estimate (19) is also a consequence of Proposition 2.8 below. The constant  $C$  coming from this Proposition could may be different from the one in [5]. When Proposition 2.8 is used, the constant  $C$  depends on the  $L^2(\mathbb{R}^3)$  norms of the functions  $\chi(|k|)|k|^{1/2}$  and  $\chi(|k|)|k|^{-1/2}$ .

The aim of this section is to establish an asymptotic expansion as  $h$  tends to 0 of the eigenvalue  $E_h$  and of an unitary eigenvector  $\varphi_h$ .

## 2.1 Statement.

**Theorem 2.2.** *Suppose that there exists  $\rho > 0$  such that the function  $\chi$  in (5) and (6) vanishes for  $|k| \leq \rho$ . Assume that  $\beta \neq 0$ . Let  $E_h$  be the infimum of the spectrum of  $H(h)$  and let  $\varphi_h$  be a corresponding normalized eigenvector of  $H(h)$ . Then, there exists a sequence of elements of  $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$  denoted  $u_j$  ( $j \geq 0$ ) and a sequence of real numbers  $\lambda_j$  ( $j \geq 1$ ) such that,*

(i) *One has,*

$$u_0 = \Psi_0 \otimes a_\emptyset, \quad \lambda_1 = -N|\beta|. \quad (20)$$

(ii) *For all integers  $p$ , there exists  $C_p$  satisfying for  $h$  small enough,*

$$\left| E_h - \sum_{j=0}^p \lambda_j h^j \right| \leq C h^{p+1}. \quad (21)$$

(iii) *Moreover, for all integers  $p$  and any  $h > 0$ , there exists  $\rho_h > 0$  and  $\theta_h$  (depending on  $h$  and  $m$ ) such that, setting*

$$V_{2p+1}(h) = \sum_{j=0}^{2p+1} u_j h^{j/2} - \rho_h e^{i\theta_h} \varphi_h, \quad (22)$$

*we have for  $h$  sufficiently small,*

$$\|V_{2p+1}(h)\| \leq C_p h^{p+1}, \quad (23)$$

*where the constant  $C_p$  is independent on  $h$ .*

(iv) *We also have,*

$$\langle (N \otimes I) V_{2p+1}(h), V_{2p+1}(h) \rangle \leq C_p h^{2p+1}. \quad (24)$$

## 2.2 Formal construction of the expansion.

*Additional details on the Fock space.* Let us recall that,

$$\mathcal{F}_s(H_{\mathbb{C}}) = \oplus_{m \geq 0} \mathcal{F}_m, \quad (25)$$

where  $\mathcal{F}_0 = \mathbb{C}$  and  $\mathcal{F}_m$  is completion of the  $m$ -fold symmetric tensor product  $H_{\mathbb{C}} \odot \cdots \odot H_{\mathbb{C}}$ . One may then consider an element of  $\mathcal{F}_m$  as a symmetric map  $f$  from  $(\mathbb{R}^3)^m$  to  $(\mathbb{C}^3)^{\otimes m}$  satisfying for all  $a_2, \dots, a_m$  in  $\{1, 2, 3\}$  and for all  $k_1, \dots, k_m$  in  $\mathbb{R}^3$ ,

$$\sum_{j=1}^3 k_{1,j} f_{j,a_2,\dots,a_m}(k_1, \dots, k_m) = 0. \quad (26)$$

We use here the notation  $k_1 = (k_{1,1}, k_{1,2}, k_{1,3})$ . In addition, the components of this function  $f$  should be in  $L^2(\mathbb{R}^{3m})$ , which is defining the norm in  $\mathcal{F}_m$ . Thus, an element of  $\mathcal{F}_s(H_{\mathbb{C}})$  is a sequence  $f = (f_m)_{(m \geq 0)}$  where  $f_m$  is an element of  $\mathcal{F}_m$  and one has,

$$\|f\|^2 = \sum_{m \geq 0} \|f_m\|^2. \quad (27)$$

We shall denote by  $\Psi_0$  a unitary element of  $\mathcal{F}_0$ .

For any  $\rho > 0$  and  $m \geq 1$ ,  $\mathcal{F}_m(\rho)$  stands for the set of elements  $f$  in  $\mathcal{F}_m$  satisfying  $f(k_1, \dots, k_m)$  belongs to  $\mathcal{S}(\mathbb{R}^{3m})$  and is vanishing if one of the  $|k_j|$  is  $\leq \rho$ . If  $m = 0$ , it is agreed that  $\mathcal{F}_0(\rho) = \mathcal{F}_0$ . It is also agreed that  $\mathcal{F}_m = 0$  if  $m < 0$ . One sets,

$$\mathcal{F}_{even}(\rho) = \mathcal{F}_0 \oplus \mathcal{F}_2(\rho) \oplus \mathcal{F}_4(\rho) \oplus \dots \quad (28)$$

$$\mathcal{F}_{odd}(\rho) = \mathcal{F}_1(\rho) \oplus \mathcal{F}_3(\rho) \oplus \mathcal{F}_5(\rho) \oplus \dots \quad (29)$$

Elements in these spaces here are finite sums.

Let us recall that, if  $M : H \rightarrow H$  is the multiplication by  $\omega(k) = |k|$  and if  $f$  is a rapidly decreasing function in  $\mathcal{F}_m$  then one has,

$$(\mathrm{d}\Gamma(M)f)(k_1, \dots, k_m) = (|k_1| + \dots + |k_m|)f(k_1, \dots, k_m). \quad (30)$$

We remind that  $H_{ph} = h\mathrm{d}\Gamma(M)$ . For all  $x \in \mathbb{R}^3$  and for each  $m \leq 3$ , the operator  $B_m(x)$  corresponding to the semiclassical parameter  $h$  is defined by (4). Therefore, we can write,

$$H(h) = hK_1 + h^{3/2}K_{3/2}, \quad (31)$$

with

$$K_1 = \mathrm{d}\Gamma(M) \otimes I + I \otimes T_0, \quad T_0 = \sum_{\lambda=1}^N \sum_{m=1}^3 \beta_m \sigma_m^{[\lambda]}, \quad (32)$$

$$K_{3/2} = \sum_{\lambda=1}^N \sum_{m=1}^3 \Phi_S(a_m(x_\lambda) + ib_m(x_\lambda)) \otimes \sigma_m^{[\lambda]}, \quad (33)$$

where  $a_m(x_\lambda)$  and  $b_m(x_\lambda)$  are defined in (5) and (6). The operators  $K_1$  and  $K_{3/2}$  are independent on  $h$ .

With these notations, the formal construction of the asymptotic expansion is provided by the following Proposition.

**Proposition 2.3.** *Let  $\rho > 0$  be such that the function  $\chi$  in (5)(6) is vanishing for  $|k| \leq \rho$ . Suppose  $\beta \neq 0$ . Then, there exists a sequence of elements in  $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$  denoted by  $u_j$  ( $j \geq 0$ ) and a sequence of real numbers  $\lambda_j$  ( $j \geq 1$ ) such that,  $u_0$  and  $\lambda_1$  are given in (20) and if  $j$  is even,*

$$u_j \in \mathcal{F}_{even}^{(j)}(\rho) \otimes \mathcal{H}_{sp}, \quad (34)$$

*and if  $j$  is odd,*

$$u_j \in \mathcal{F}_{odd}^{(j)}(\rho) \otimes \mathcal{H}_{sp} \quad (35)$$

*and such that, for all integers  $p$ , setting,*

$$U^{(p)}(h) = \sum_{j=0}^p u_j h^{j/2}, \quad \lambda^{(p)}(h) = \lambda_1 h + \cdots + \lambda_p h^p, \quad (36)$$

*we have*

$$\left( H(h) - \lambda^{(p+1)}(h) \right) U^{(2p)}(h) = R^{(2p)}(h), \quad (37)$$

$$\left( H(h) - \lambda^{(p+1)}(h) \right) U^{(2p+1)}(h) = R^{(2p+1)}(h), \quad (38)$$

*where the  $R^{(j)}(h)$  are expressed as following,*

$$R^{(2p)}(h) = \sum_{k \geq 1} h^{p+1+(k/2)} f_{2p}^{(p+1+(k/2))}, \quad (39)$$

*with the  $f_{2p}^{(p+1+(k/2))}$  being elements of  $\mathcal{F}_s(H_{\mathbb{C}}) \otimes \mathcal{H}_{sp}$ ,*

$$R^{(2p+1)}(h) = \sum_{k \geq 0} h^{p+2+(k/2)} f_{2p+1}^{(p+2+(k/2))}, \quad (40)$$

*with the  $f_{2p+1}^{(p+2+(k/2))}$  belonging to  $\mathcal{F}_s(H_{\mathbb{C}}) \otimes \mathcal{H}_{sp}$ . The sums in the right hand sides of (39) and (40) are finite.*

The above elements  $u_j$  are independent on  $h$ . The proof uses the following Lemma.

**Lemma 2.4.** *Let  $\lambda_1$  be defined in (20), and  $T_0$  in (32). Then, for all  $f$  in  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$ , (resp. in  $\mathcal{F}_{even}(\rho) \otimes \mathcal{H}_{sp}$ ), there exists  $u$  in  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$ , (resp. in  $\mathcal{F}_{even}(\rho) \otimes \mathcal{H}_{sp}$ ) satisfying,*

$$(d\Gamma(M) \otimes I + I \otimes (T_0 - \lambda_1))u = f - \Pi f, \quad (41)$$

*where  $\Pi$  is the orthogonal projection on  $u_0 = \Psi_0 \otimes a_{\emptyset}$ .*



Note that  $\Pi f = 0$  if  $f$  is in  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$ .

*Proof of the Lemma.* One may write,

$$f = \sum_{E \subset \{1, \dots, N\}} \sum_{m \geq 0} f_{E,m} \otimes a_E,$$

with  $f_{E,m}$  in  $\mathcal{F}_m(\rho)$ , vanishing for even  $m$  (resp. for odd  $m$ ). If  $m \geq 1$ , set

$$u_{E,m}(k_1, \dots, k_m) = \frac{f_{E,m}(k_1, \dots, k_m)}{|k_1| + \dots + |k_m| + 2|\beta||E|}.$$

Since  $f_{E,m}$  vanishes in neighborhood of the origin, this element is well defined, even when  $E$  is empty. If  $m = 0$  and  $E \neq \emptyset$ , set

$$u_{E,0} = \frac{f_{E,0}}{2|\beta||E|}.$$

Then set,

$$u = \sum_{m \geq 1} \sum_{E \subset \{1, \dots, N\}} u_{E,m} \otimes a_E + \sum_{\substack{E \subset \{1, \dots, N\} \\ E \neq \emptyset}} u_{E,0} \otimes a_E.$$

This element  $u$  has the stated properties in the Lemma.

□

*Proof of Proposition 2.3.* Note that the operator  $K_1$  defined in (32) maps each of the two spaces  $\mathcal{F}_{even}(\rho) \otimes \mathcal{H}_{ph}$  and  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{ph}$  into itself, whereas  $K_{3/2}$  defined in (33) maps each of these two spaces into each other. This comes from the Segal field  $\Phi_S$  definition (see [12]). By writing that the coefficient of  $h^j$  ( $j \leq p+1$ ) in the left hand side of (37) or (38) is zero, one sees that the  $u_j$  and the  $\lambda_j$  have to satisfy the following relations,

$$(K_1 - \lambda_1)u_0 = 0, \tag{42}$$

$$(K_1 - \lambda_1)u_1 + K_{3/2}u_0 = 0, \tag{43}$$

$$(K_1 - \lambda_1)u_2 + K_{3/2}u_1 - \lambda_2u_0 = 0, \tag{44}$$

$$(K_1 - \lambda_1)u_3 + K_{3/2}u_2 - \lambda_2u_1 = 0. \tag{45}$$

More generally, if  $m = 2p$  is even, one needs,

$$(K_1 - \lambda_1)u_{2p} + K_{3/2}u_{2p-1} - \lambda_2u_{2p-2} - \dots - \lambda_{p+1}u_0 = 0 \tag{46}$$

and if  $m = 2p+1$  is odd,

$$(K_1 - \lambda_1)u_{2p+1} + K_{3/2}u_{2p} - \lambda_2u_{2p-1} - \dots - \lambda_{p+1}u_1 = 0. \tag{47}$$

One has,  $d\Gamma(M)\Psi_0 = 0$  and  $(T_0 - \lambda_1)a_\emptyset = 0$ , thus, the elements  $u_0$  and  $\lambda_1$  defined in (20) satisfy (42). Since the operator  $K_{3/2}$  exchanges parity,  $K_{3/2}u_0$  is in  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$ . According to the Lemma 2.4, there exists  $u_1$  in  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$  satisfying (43). Set  $p \geq 0$ . Suppose that  $u_0, \dots, u_{2p+1}$  and  $\lambda_1, \dots, \lambda_{p+1}$  satisfying (46) and (47) are already constructed. In order to determine  $u_{2p+2}$  and  $\lambda_{p+2}$ , one applies Lemma 2.4 with,

$$f_{2p+2} = \begin{cases} -K_{3/2}u_{2p+1} + \lambda_2 u_{2p} + \dots + \lambda_{p+1} u_2 & \text{if } p \geq 1 \\ -K_{3/2}u_1 & \text{if } p = 0 \end{cases}. \quad (48)$$

Since  $K_{3/2}$  exchanges parity, this element belongs to  $\mathcal{F}_{even}(\rho) \otimes \mathcal{H}_{sp}$ . One defines  $\lambda_{p+2}$  by,

$$\lambda_{p+2} = - \langle f_{2p+2}, u_0 \rangle. \quad (49)$$

According to Lemma 2.4, there exists  $u_{2p+2}$  in  $\mathcal{F}_{even}(\rho) \otimes \mathcal{H}_{sp}$  such that,

$$(K_1 - \lambda_1)u_{2p+2} = f_{2p+2} + \lambda_{p+2}u_0,$$

that is to say, (46) with  $p$  replaced by  $p + 1$ . To get  $u_{2p+3}$ , Lemma 2.4 is applied with,

$$f_{2p+3} = -K_{3/2}u_{2p+2} + \lambda_2 u_{2p+1} + \dots + \lambda_{p+2}u_1.$$

This element belongs to  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$  and consequently,  $\Pi f_{2p+3} = 0$ . According to Lemma 2.4, there indeed exists  $u_{2p+3}$  in  $\mathcal{F}_{odd}(\rho) \otimes \mathcal{H}_{sp}$  satisfying

$$(K_1 - \lambda_1)u_{2p+3} = f_{2p+3}.$$

We have therefore constructed the sequences  $(u_j)$  and  $(\lambda_j)$  satisfying (46) and (47). The properties in the statement of the Proposition then follows. □

The elements  $u_0$  and  $\lambda_1$  are defined in (20). The following Proposition gives an explicit computation of  $u_1$  and  $\lambda_2$ . One sees that here, we do not need any hypothesis on the behaviour of  $\chi$  in a neighborhood of the origin.

**Proposition 2.5.** *We have,*

$$\lambda_2 = -NC - \frac{1}{2} \sum_{\lambda, \mu \leq N} F(x_\lambda - x_\mu), \quad (50)$$

where  $F$  is the semiclassical parallel spins interaction function defined in (14) and

$$C = \frac{1}{2}(2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\chi(|k|)^2 |k|}{|k| + 2|\beta|} \frac{|k|^2 + k_3^2}{|k|^2} dk. \quad (51)$$

*Proof.* Let us first precise the computation of  $u_1$ . We assume that  $\beta = (0, 0, |\beta|)$ . According to (33),

$$K_{3/2}u_0 = f_\emptyset \otimes a_\emptyset + \sum_{\mu=1}^N f_\mu \otimes a_{\{\mu\}}.$$

Since  $\sigma_3^{[\mu]}a_\emptyset = -a_\emptyset$ , we have,

$$f_\emptyset = - \sum_{\mu \leq N} \Phi_S(a_3(x_\mu) + ib_3(x_\mu))\Psi_0. \quad (52)$$

Similarly, since  $\sigma_1^{[\mu]}a_\emptyset = a_{\{\mu\}}$  and  $\sigma_2^{[\mu]}a_\emptyset = -ia_{\{\mu\}}$ ,

$$f_\mu = \Phi_S(a_1(x_\mu) + ib_1(x_\mu))\Psi_0 - i\Phi_S(a_2(x_\mu) + ib_2(x_\mu))\Psi_0. \quad (53)$$

All these elements are in  $\mathcal{F}_1 = H_{\mathbb{C}}$ . For all  $X$  in  $H_{\mathbb{C}}$ , we can identify  $\Phi_S(X)\Psi_0$  with  $X/\sqrt{2}$  which is therefore a function of  $k \in \mathbb{R}^3$ . The element  $u_1$  needs to satisfy (43). In view of Lemma 2.4, it can be written as,

$$u_1 = u_\emptyset \otimes a_\emptyset + \sum_{\mu=1}^N u_\mu \otimes a_{\{\mu\}}, \quad (54)$$

where  $u_\emptyset$  and the  $u_\mu$  are in  $\mathcal{F}_1$ , defined by,

$$u_\emptyset(k) = -\frac{f_\emptyset(k)}{|k|}, \quad u_\lambda(k) = -\frac{f_\lambda(k)}{|k| + 2|\beta|}. \quad (55)$$

According to (49) and (48) (with  $p = 0$ ), we have,

$$\lambda_2 = \langle K_{3/2}u_1, u_0 \rangle = \langle u_1, K_{3/2}u_0 \rangle. \quad (56)$$

Consequently,

$$\lambda_2 = \langle u_\emptyset, f_\emptyset \rangle + \sum_{\lambda=1}^N \langle u_\lambda, f_\lambda \rangle.$$

We have,

$$\langle u_\emptyset, f_\emptyset \rangle = -\frac{1}{2} \sum_{\lambda, \mu \leq N} \int_{\mathbb{R}^3} \left( a_3(x_\lambda) + ib_3(x_\lambda) \right)(k) \cdot \left( a_3(x_\mu) - ib_3(x_\mu) \right)(k) \frac{dk}{|k|}.$$

Therefore, using (5) and (6),

$$\langle u_\emptyset, f_\emptyset \rangle = -\frac{1}{2} \sum_{\lambda, \mu \leq N} F(x_\lambda - x_\mu),$$

where  $F$  is the semiclassical parallel spins interaction function defined in (14). We similarly see that,

$$\langle u_\lambda, f_\lambda \rangle = -\frac{1}{2} \int_{\mathbb{R}^3} \left| (a_1(x_\lambda) + ib_1(x_\lambda)) - i(a_2(x_\lambda) + ib_2(x_\lambda)) \right|^2 \frac{dk}{|k| + 2|\beta|}.$$

One again uses (5) and (6) noticing that,  $|(k \wedge e_1) - i(k \wedge e_2)|^2 = |k|^2 + k_3^2$ . Consequently,

$$\langle u_\lambda, f_\lambda \rangle = -C$$

where  $C$  is defined in (51).

□

For the interpretation of the function  $F$ , we note that the classical potential vector associated to the current density  $\mathbf{J}(x) = \mathbf{n}_\beta \wedge \text{grad}\rho(x)$ , where  $\rho$  is defined in (16), is

$$\mathbf{A}^{class}(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} |\chi(|k|)|^2 \sin(k \cdot x) \frac{k \wedge \mathbf{n}_\beta}{|k|^2} dk.$$

The magnetic field is  $\mathbf{B}^{class} = \text{rot}\mathbf{A}^{class}$ . In particular, its projection on the direction of  $\beta$  is  $F(x)$ , with  $F$  defined in (14). By translation,  $F(x_\lambda - x_\mu)$  is the classical magnetic field created by the spin centered at  $x_\lambda$  (that is to say, by the current density  $\mathbf{n}_\beta \wedge \text{grad}\rho(x - x_\lambda)$ ), taken at  $x_\mu$  and projected on the direction where all the spins are aligned (generated by  $\mathbf{n}_\beta$ ). According to the coupling constants, one can think that  $-h^2 F(x_\lambda - x_\mu)$  is the interaction energy of the spins centered at  $x_\lambda$  and  $x_\mu$ , aligned and pointing in the same direction parallel to  $\mathbf{n}_\beta$ .

We shall now compute the average magnetic field taken on the ground state first order asymptotic expansion. We do not have any hypothesis on the behaviour of  $\chi$  at the origin.

**Proposition 2.6.** *We have, for any  $x \in \mathbb{R}^3$ ,*

$$< (\mathbf{B}(x) \otimes I)(u_0 + h^{1/2}u_1), (u_0 + h^{1/2}u_1) > = \mathbf{B}^{class}(x), \quad (57)$$

$$< (\mathbf{E}(x) \otimes I)(u_0 + h^{1/2}u_1), (u_0 + h^{1/2}u_1) > = 0, \quad (58)$$

where  $\mathbf{B}^{class}(x)$  is defined in Section 1.

*Proof.* One can suppose that  $\beta = (0, 0, |\beta|)$ . The above computations show that the classical magnetic associated to the current density defined in (15) with  $n_\beta = e_3 = (0, 0, 1)$ , is

$$B_m^{class}(x) = h(2\pi)^{-3} \sum_{\lambda=1}^N \int_{\mathbb{R}^3} |\chi(k)|^2 \cos(k \cdot (x - x_\lambda)) \frac{(k \wedge e_m) \cdot (k \wedge e_3)}{|k|^2} dk. \quad (59)$$

Besides, we have,

$$< (B_m(x) \otimes I)(u_0 + h^{1/2}u_1), (u_0 + h^{1/2}u_1) > = 2h^{1/2} \text{Re} < (B_m(x) \otimes I)u_0, u_1 > .$$

Indeed, for all  $u$  belonging to one of the  $\mathcal{F}_j$ , we have  $< B_m(x)u, u > = 0$ . According to the construction (54) and (55) of  $u_1$  in the proof of Proposition 2.5, we have,

$$< (B_m(x) \otimes I)u_0, u_1 > = < B_m(x)\Psi_0, u_\emptyset > .$$

Using the expression (55) of  $u_\emptyset$  and next, the one of  $f_\emptyset$  in (52), both considered as elements of  $H_{\mathbb{C}}$ , we obtain,

$$\begin{aligned} \langle (B_m(x) \otimes I)u_0, u_1 \rangle &= \int_{\mathbb{R}^3} \left( a_m(x) + ib_m(x) \right)(k) \cdot \frac{\overline{f_\emptyset(k)}}{|k|} dk \\ &= \sum_{\mu \leq N} \int_{\mathbb{R}^3} \left( a_m(x) + ib_m(x) \right)(k) \cdot \frac{(a_3(x_\mu) - ib_3(x_\mu))(k)}{|k|} dk. \end{aligned}$$

One therefore recovers the right hand side of (59).

□

### 2.3 Control of the remainder term.

The control of the error terms in Theorem 2.2, that is to say, points (21), (23) and (24), are a consequence of the following Theorem together with the construction in Proposition 2.3.

Set  $L$  a selfadjoint extension in  $H$  of the operator,

$$L = \frac{1}{i} \frac{\partial}{\partial r} + \frac{1}{ir}, \quad (60)$$

where  $r = |k|$ .

**Theorem 2.7.** *Let  $U_h$  be an element of  $D(H(h))$ , satisfying,*

$$H(h)U_h = \lambda(h)U_h + R_h. \quad (61)$$

*Suppose that there are  $C > 0$  and  $p \geq 1$  such that, for all  $h$  in  $(0, 1)$ ,*

$$\|U_h - (\Psi_0 \otimes a_\emptyset)\| \leq Ch^{1/2}. \quad (62)$$

*Suppose also that  $U_h$  and  $R_h$  are in the domain of  $d\Gamma(L) \otimes I$ , and that:*

$$\|U_h\| + \|(d\Gamma(L) \otimes I)U_h\| \leq C, \quad \|R_h\| + \|(d\Gamma(L) \otimes I)R_h\| \leq Ch^{p+2}. \quad (63)$$

*Then, for any sufficiently small  $h$ ,*

$$|\lambda(h) - E_h| \leq 2Ch^{p+2}. \quad (64)$$

*Set  $\varphi_h$  a normalized eigenvector corresponding to the eigenvalue  $E_h$ , the infimum of the spectrum of  $H(h)$ , satisfying (19). From (19), for  $h$  small enough, one can choose  $\rho_h > 0$  and  $\theta_h$  such that the function  $V_h$  defined by,*

$$V_h = U_h - \rho_h e^{i\theta_h} \varphi_h, \quad (65)$$

satisfies,

$$\langle V_h, \Psi_0 \otimes a_\emptyset \rangle = 0. \quad (66)$$

The function  $V_h$  then satisfies,

$$\|V_h\| \leq Ch^{p+1}. \quad (67)$$

We also have for small enough  $h$ ,

$$\langle (N \otimes I)V_h, V_h \rangle \leq Kh^{2p}. \quad (68)$$

*Proof of (64).* In view of (19) and (62), one deduces,

$$\|\varphi_h - U_h\| \leq Ch^{1/2}.$$

As a consequence, for small enough  $h$ ,

$$|\langle U_h, \varphi_h \rangle| \geq 1/2.$$

By equaling the scalar products of the two hand sides of (61) with  $\varphi_h$  which satisfies  $H(h)\varphi_h = E_h\varphi_h$ , one obtains,

$$|E_h - \lambda(h)| |\langle U_h, \varphi_h \rangle| \leq \|R_h\| \leq Ch^{p+2}.$$

For  $h$  small enough, inequality (64) then follows.

□

Estimates (67) and (68) are a consequence of the two following Propositions. The first one is relying on a conjugated operator argument.

**Proposition 2.8.** *Let  $V_h$  be an element of  $D(H(h))$  and  $f_h$  be element of  $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$  satisfying,*

$$(H(h) - E_h)V_h = f_h, \quad (69)$$

where  $E_h \leq -N|\beta|h$ . Suppose that  $f_h$  belongs to the domain of  $d\Gamma(L) \otimes I$ , where  $L$  a selfadjoint extension in  $H$  of the operator (60). We suppose also that (66) is satisfied, and that

$$16h^{1/2} \sum_{\lambda=1}^N \sum_{m=1}^3 |LA_m(x_\lambda)| \leq 1 \quad 4 \frac{h^{1/2}}{|\beta|} 2^{1+(N/2)} \sum_{\lambda=1}^N \sum_{m=1}^3 |A_m(x_\lambda)| \leq 1. \quad (70)$$

Then we have:

$$\|V_h\| \leq \frac{16}{h} \|(d\Gamma(L) \otimes I)f_h\| + \frac{4}{|\beta|h} \|f_h\|, \quad (71)$$

$$\langle (N \otimes I)V_h, V_h \rangle \leq \frac{300}{h^2} \|(d\Gamma(L) \otimes I)f_h\|^2 + \frac{32}{h^2|\beta|^2} \|f_h\|^2. \quad (72)$$

*Proof. First step.* When  $M$  is the multiplication by  $|k|$ , and  $L$  is defined in (60), we have,

$$[d\Gamma(L), d\Gamma(M)] = d\Gamma([L, M]) = \frac{1}{i}d\Gamma(I) = \frac{1}{i}N.$$

If  $V_h$  satisfies (69), we have:

$$\text{Im} \langle (H(h) - E_h)V_h, (d\Gamma(L) \otimes I)V_h \rangle = \text{Im} \langle f_h, (d\Gamma(L) \otimes I)V_h \rangle.$$

We use the notations (31), (32), (33) for the operator  $H(h)$ . Recalling that  $H_{ph} = h d\Gamma(M)$ , it follows from the above commutator relation that:

$$\text{Im} \langle (H_{ph} \otimes I)V_h, (d\Gamma(L) \otimes I)V_h \rangle = -\frac{h}{2} \langle (N \otimes I)V_h, V_h \rangle$$

We have

$$\text{Im} \langle ((I \otimes T_0) - E_h)V_h, (d\Gamma(L) \otimes I)V_h \rangle = 0$$

Therefore:

$$-\frac{h}{2} \langle (N \otimes I)V_h, V_h \rangle + h^{3/2} \text{Im} \langle K_{3/2}V_h, (d\Gamma(L) \otimes I)V_h \rangle = \text{Im} \langle f_h, (d\Gamma(L) \otimes I)V_h \rangle$$

Setting  $A_m(x_\lambda) = a_m(x_\lambda) + ib_m(x_\lambda)$ , and  $L$  is defined in (60), we have classically:

$$i[d\Gamma(L), \Phi_S(A_m(x_\lambda))] = \Phi_S(iLA_m(x_\lambda))$$

By (7), we have:

$$\|(\Phi_S(iLA_m(x_\lambda)) \otimes \sigma_m^{[\lambda]})V_h\|^2 \leq |LA_m(x_\lambda)|^2 \left[ \|V_h\|^2 + \langle (N \otimes I)V_h, V_h \rangle \right].$$

Therefore:

$$\begin{aligned} \frac{h}{2} \langle (N \otimes I)V_h, V_h \rangle &\leq \frac{h^{3/2}}{2} \|V_h\| \left[ \|V_h\|^2 + \langle (N \otimes I)V_h, V_h \rangle \right]^{1/2} \sum_{\lambda=1}^N \sum_{m=1}^3 |LA_m(x_\lambda)| + \dots \\ &\dots + \|(d\Gamma(L) \otimes I)f_h\| \|V_h\| \end{aligned}$$

If (70) is satisfied, then we have:

$$\langle (N \otimes I)V_h, V_h \rangle \leq \frac{4}{h} \|(d\Gamma(L) \otimes I)f_h\| \|V_h\| + \frac{1}{16} \|V_h\|^2. \quad (73)$$

*Second step.* Let us denote by  $P_\emptyset$  the projection in  $\mathcal{H}_{sp}$  on the vectorial line generated by  $a_\emptyset$  and by  $P_\emptyset^\perp$  the projection on the orthogonal subspace. Also,  $P_\Omega$  denotes the projection in  $\mathcal{H}_{ph}$  on the vacuum  $\Psi_0$  and  $P_\Omega^\perp$  stands for the projection on the orthogonal subspace. By (69), we have:

$$\langle (P_\Omega \otimes P_\emptyset^\perp)(H(h) - E_h)V_h, V_h \rangle = \langle (P_\Omega \otimes P_\emptyset^\perp)f_h, V_h \rangle$$

We have  $P_\Omega H_{ph} = 0$ . Note that  $P_\emptyset^\perp T_0 \geq |\beta|(1 - N)P_\emptyset^\perp$ . In particular, if  $E_h \leq -N|\beta|h$ ,

$$\langle (P_\Omega \otimes P_\emptyset^\perp)(I \otimes hT_0 - E_h) \rangle \geq h|\beta| \langle (P_\Omega \otimes P_\emptyset^\perp)V_h, V_h \rangle.$$

Therefore:

$$h|\beta| \langle (P_\Omega \otimes P_\emptyset^\perp)V_h, V_h \rangle \leq h^{3/2} \langle (P_\Omega \otimes P_\emptyset^\perp)K_{3/2}V_h, V_h \rangle + \langle (P_\Omega \otimes P_\emptyset^\perp)f_h, V_h \rangle$$

We have:

$$\begin{aligned} |\langle (P_\Omega \otimes P_\emptyset^\perp)K_{3/2}V_h, V_h \rangle| &\leq \sum_{E \neq \emptyset} |\langle V_h, K_{3/2}(\Psi_0 \otimes a_E) \rangle| |\langle V_h, (\Psi_0 \otimes a_E) \rangle| \\ &\leq M \|(P_\Omega \otimes P_\emptyset^\perp)V_h\| \|V_h\| \end{aligned}$$

with:

$$M = \left[ \sum_{E \neq \emptyset} \|K_{3/2}(\Psi_0 \otimes a_E)\|^2 \right]^{1/2} \leq 2^{N/2} \sum_{\lambda=1}^N \sum_{m=1}^3 \|\Phi_S(A_m(x_\lambda))\Psi_0\|$$

By (7)

$$\|\Phi_S(A_m(x_\lambda))\Psi_0\| \leq 2|A_m(x_\lambda)|$$

Therefore, if (70) is satisfied:

$$\|(P_\Omega \otimes P_\emptyset^\perp)V_h\| \leq \frac{1}{16}\|V_h\| + \frac{1}{h|\beta|}\|f_h\|. \quad (74)$$

*Third step.* By the condition (66), we have:

$$\|V_h\| \leq \|(P_\Omega^\perp \otimes I)V_h\| + \|(P_\Omega \otimes P_\emptyset^\perp)V_h\|$$

$$\leq \langle (N \otimes I)V_h, V_h \rangle^{1/2} + \|(P_\Omega \otimes P_\emptyset^\perp)V_h\|$$

and therefore (71) follows from (73) and (74), and (72) follows from (71) and (73).

*End of the proof of Theorem 2.7.* Estimate (64) is already proved. If  $U_h$  satisfies (61), then we have,

$$H(h)U_h = E_h U_h + f_h, \quad f_h = R_h + (\lambda(h) - E_h)U_h. \quad (75)$$

For any small enough  $h$ , we can choose  $\rho_h > 0$  and  $\theta_h$  such that, the function  $V_h$  defined by (65) satisfies (66) and also (69). The functions  $R_h$  and  $R_h$ , and therefore  $f_h$ , are in the domain of  $d\Gamma(L) \otimes I$ . Therefore, by Proposition 2.8, if the conditions (70) are satisfied, then the estimates (71) and (72) are satisfied. By (63) and (64), we have:

$$\|f_h\| + \|(d\Gamma(L) \otimes I)f_h\| \leq Ch^{p+2}.$$



The estimates (67) and (68) follow from (71), (72) and the above inequality.

□

*End of the proof of Theorem 2.2.* We apply Theorem 2.7 with the elements  $U(h) = U^{(2p+1)}(h)$  and  $R(h) = R^{(2p+1)}(h)$  and with the real number  $\lambda(h) = \lambda^{(p+1)}(h)$  of Proposition 2.3. These elements satisfy (61) from (19). The condition (62) comes from the fact that  $u_0$  is defined in (20) and that the other  $u_j$  are independent on  $h$ . The assumption (63) comes from the fact that the  $u_j$  and the  $f_{2p+1}^{(p+2+(k/2))}$  of Proposition 2.3 are finite sums of terms all belonging to the spaces  $\mathcal{F}_m(\rho)$ . Note that  $\mathcal{F}_m(\rho)$  is invariant by the operator  $d\Gamma(L)$  where  $L$  is defined in (60). The hypotheses of Theorem 2.7 are satisfied. Inequality (21) follows from (64). Inequality (23) is a consequence of (67) and inequality (24) comes from (68).

*Proof of Theorem 1.1.* The point i) follows from inequality (21) of theorem 2.2. For the point ii), without any hypothesis on  $\chi(0)$ , the elements  $u_0$  and  $\lambda_1$  are defined by (20), and  $u_1$  and  $\lambda_2$  are constructed in Proposition 2.5. Let us prove that the construction of  $u_2$  is also possible without any hypothesis on  $\chi(0)$ . Let us now define  $u_2$  which has to satisfy (44), that is to say, taking into account the choice of  $\lambda_2$  in (56),

$$(K_1 - \lambda_1)u_2 = -(I - \Pi)K_{3/2}u_1,$$

where  $\Pi$  is the orthogonal projection on the vectorial line generated by  $u_0 = \Psi_0 \otimes a_\emptyset$ . Since  $u_1$  is defined in (54), we can write

$$K_{3/2}u_1 = \sum_{E \subset \{1, \dots, N\}} (f_E + g_E) \otimes a_E,$$

where the  $f_E$  belong to  $\mathcal{F}_0$  and the  $g_E$  lie in  $\mathcal{F}_2$ . We have  $\Pi K_{3/2}u_1 = f_\emptyset \otimes a_\emptyset$ . Consequently,

$$u_2 = \sum_{E \subset \{1, \dots, N\}} (u_E + v_E) \otimes a_E, \quad (76)$$

where the  $u_E$  are in  $\mathcal{F}_0$  and the  $v_E$  in  $\mathcal{F}_2$ . We shall have  $u_\emptyset = 0$ . If  $E$  is non empty then we shall obtain, according to Lemma 7.4,  $u_E = f_E/(2|\beta||E|)$ . The elements  $g_E$  and  $v_E$  being identified with symmetric functions on  $\mathbb{R}^3 \times \mathbb{R}^3$  and taking vector values, we have,

$$v_E(k_1, k_2) = \frac{g_E(k_1, k_2)}{|k_1| + |k_2| + 2|\beta||E|}. \quad (77)$$

According to (52)-(55), the functions  $g_E$  are linear combinations of products of the form  $(a_m(x_\lambda) + ib_m(x_\lambda))(k_1)u_\mu(k_2)$ , of products where the second factor is  $u_\emptyset(k_2)$ , and of products where the factors are exchanged. The  $u_\mu$  and  $u_\emptyset$  are rapidly decreasing at infinity and, when  $k$  tends to 0,  $u_\mu(k) = \mathcal{O}(|k|^{1/2})$  and  $u_\emptyset(k) = \mathcal{O}(|k|^{-1/2})$ . Concerning elements  $(a_m(x_\lambda) + ib_m(x_\lambda))(k)$ , they are an  $\mathcal{O}(|k|^{1/2})$  when  $k$  tends to 0. Consequently, equalities (77) therefore define elements  $v_E$  of  $\mathcal{F}_2$ , and (76) indeed defines an

element  $u_2$  of  $(\mathcal{F}_0 \oplus \mathcal{F}_2) \otimes \mathcal{H}_{sp}$  (Without vanishing assumptions on  $\chi$  at the origin, it does not seem possible to further follow the expansion). The element  $K_{3/2}u_2$  is well defined since  $K_{3/2}$  is continuous from  $(\mathcal{F}_0 \oplus \mathcal{F}_2) \otimes \mathcal{H}_{sp}$  into  $(\mathcal{F}_1 \oplus \mathcal{F}_3) \otimes \mathcal{H}_{sp}$ . We then have,

$$(H(h) - \lambda_1 h - \lambda_2 h^2) (u_0 + h^{1/2}u_1 + hu_2) = R^{(2)}(h) = h^{5/2}(K_{3/2}u_2 - \lambda_2 u_1) - h^3 \lambda_2 u_0.$$

Thus, we have  $\|R^{(2)}(h)\| \leq Ch^{5/2}$ . Taking the scalar products of both sides with  $\varphi_h$  satisfying (19) and  $(H(h) - E_h)\varphi_h = 0$ , one therefore obtains estimate (13).

□

*Proof of Theorem 1.2.* Without any hypothesis on  $\chi(0)$ , we determined  $u_0$  and  $\lambda_1$  from (20), and  $u_1$  according to Proposition 2.5. One can choose  $\rho_h$  and  $\theta_h$  such that the following function,

$$V_h = u_0 + h^{1/2}u_1 - \rho_h e^{i\theta_h} \varphi_h$$

satisfies (66). According to Proposition 2.6, we have,

$$\mathbf{B}^{class}(x) = \langle (\mathbf{B}(x) \otimes I)(u_0 + h^{1/2}u_1), (u_0 + h^{1/2}u_1) \rangle.$$

Thus,

$$|B_m^{class}(x) - \rho_h^2 \langle B_m(x)\varphi_h, \varphi_h \rangle| \leq \|B_m(x)V_h\| \left(2\rho_h + \|V_h\|^2\right). \quad (78)$$

By the construction in Proposition 2.5, especially (42) and (43), the function  $V_h$  defined above satisfies (69) with,

$$f_h = h^2 K_{3/2}u_1 + (\lambda_1 h - E_h)(u_0 + h^{1/2}u_1).$$

According to point (ii) of Theorem 1.1, we have  $|\lambda_1 h - E_h| \leq Ch^2$ . Now, if  $\chi(0) = 0$ , let us prove that  $u_1$  and  $K_{3/2}u_1$  lie in the domain of the operator  $d\Gamma(L) \otimes I$ . One follows the construction of  $u_1$  given in (54). The elements  $u_\emptyset$  and  $u_\lambda$  of  $\mathcal{F}_1$  defined in (55) can be identified to elements of  $H_{\mathbb{C}}$  and then to functions on  $\mathbb{R}^3$  and taking values in  $\mathbb{C}^3$ . In general,  $u_\emptyset(k) = \mathcal{O}(|k|^{-1/2})$  when  $k$  tends to 0 and this function is not belonging to the domain of the operator  $L$  defined in (60). However, if  $\chi(0) = 0$ , we have  $u_\emptyset(k) = \mathcal{O}(|k|^{1/2})$  when  $k$  tends to 0 and  $u_\emptyset$  lies in  $D(L)$ . This also holds true for  $u_\lambda$ . Consequently,  $u_1$  lies in the domain of the operator  $d\Gamma(L) \otimes I$ . This is also valid for  $K_{3/2}u_1$ . Therefore,  $f_h$  belongs to the domain of  $d\Gamma(L) \otimes I$  and

$$\|f_h\| + \|(d\Gamma(L) \otimes I)f_h\| \leq Ch^2.$$

Then, we can apply Proposition 2.8. This enables to write, for  $h$  small enough,

$$\|V_h\| \leq Ch, \quad \langle (N \otimes I)V_h, V_h \rangle \leq Ch^2.$$

According to (7),

$$\|B_m(x)V_h\| \leq Ch^{1/2}\|V_h\| + Ch^{1/2} \langle (N \otimes I)V_h, V_h \rangle^{1/2}.$$

Thus,  $\|B_m(x)V_h\| \leq Ch^{3/2}$ . Condition (66), Definition (20) of  $u_0$  and the property (19) for  $\varphi_h$  imply that  $|1 - \rho_h e^{i\theta_h}| \leq Ch^{1/2}$ . The right hand side of (78) is then  $\mathcal{O}(h^{3/2})$ . Since  $\mathbf{B}_m^{class}(x)$  is an  $\mathcal{O}(h)$ , the above equality (78) shows that  $\langle B_m(x)\varphi_h, \varphi_h \rangle$  is also an  $\mathcal{O}(h)$ . Consequently,

$$(\rho_h^2 - 1) \langle B_m(x)\varphi_h, \varphi_h \rangle = \mathcal{O}(h^{3/2}).$$

Theorem 1.3 then follows.

□

## References

- [1] L. Amour, L. Jager, J. Nourrigat, *On bounded Weyl pseudodifferential operators in Wiener spaces*, Journal of Functional Analysis 269 (2015), pp. 2747-2812.
- [2] L. Amour, R. Lascar, J. Nourrigat, *Beals characterization of pseudodifferential operators in Wiener spaces*, Appl. Math. Res. Express (2016).
- [3] L. Amour, R. Lascar, J. Nourrigat, *Weyl calculus in QED I. The unitary group*, preprint, arXiv:1510.05293, october 2015.
- [4] A. Arai, M. Hirokawa, *On the existence and uniqueness of ground states of a generalized spin-boson model*, J. Funct. Anal. 151 (1997), no. 2, 455–503.
- [5] V. Bach, J. Fröhlich, I. M. Sigal, *Quantum electrodynamics of confined nonrelativistic particles*, Adv. Math. 137 (1998), no. 2, 299–395.
- [6] F. Bloch, *Nuclear Induction*, Physical Review **70** 460-473, (1946).
- [7] J. Dereziński, C. Gérard, *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians*. Rev. Math. Phys. 11 (1999), no. 4, 383-450.
- [8] C. Gérard, *On the existence of ground states for massless Pauli-Fierz Hamiltonians*, Ann. Henri Poincaré **1** (2000), 443–459.
- [9] M. Hirokawa, F. Hiroshima, J. Lörinczi, *Spin-boson model through a Poisson-driven stochastic process*, Math. Z. **277** (2014), 1165–1198.

- [10] M. Hübner, H. Spohn, *Spectral properties of the spin-boson Hamiltonian.*, Annales de l'I. H. P., Section A , tome 62, **3** (1995), 289-323.
- [11] E. Lieb, M. Loss, *A note on polarization vectors in quantum electrodynamics.* Comm. Math. Phys. **252** (2004), no. 1-3, 477-483.
- [12] M. Reed, B. Simon, *Methods of modern mathematical physics*, Vol II, Fourier Analysis, selfadjointness, Academic Press, 1975.
- [13] F. A. Reuse, *Electrodynamique et Optique Quantiques*, Presses Polytechniques et Universitaires Romandes, Lausanne, 2007.

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